Reconciling Cash Flow And Share Price Volatility

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Research has shown that the volatility of unlevered share price is far greater than the volatility of the underlying cash flows. In this whitepaper we will reconcile cash flow and share price volatility.

The Equation For Company Value

We will define the variable C_t to be random annualized cash flow at time t, the variable μ to be the continuous-time secular cash flow growth rate, the variable σ_c to be the cash flow growth rate volatility, and the variable δW_t to be the change in the driving Brownian motion. The stochastic differential equation (SDE) that defines the change in annualized cash flow over time is...

$$\delta C_t = \mu\, C_t\, \delta t + \sigma_c\, C_t\, \delta W_t \text{ ...where... } \delta W_t \sim N\Big[0, \delta t\Big] \tag{1}$$

The solution to Equation (1) above is the equation for random annualized cash flow at time t. The equation for random annualized cash flow at time t (unknown) as a function of annualized cash flow at time zero (known) is...

$$C_t = C_0 \operatorname{Exp}\left\{\left(\mu - \frac{1}{2}\,\sigma_c^2\right)t + \sigma_c \sqrt{t}\, x\right\} \text{ ...where... } x \sim N\Big[0, 1\Big] \tag{2}$$

Using Equation (2) above the equation for expected annualized cash flow at time t is...

$$\mathbb{E}\Big[C_t\Big] = C_0 \operatorname{Exp}\Big\{\mu\, t\Big\} \tag{3}$$

We will define the variable V_t to be company value at time t and the variable κ to be the continuous-time cash flow discount rate. Using Equation (3) above the equation for expected company value at time t is...

$$\mathbb{E}\Big[V_t\Big] = \int_t^\infty \mathbb{E}\Big[C_s\Big] \operatorname{Exp}\Big\{-\kappa\,(s-t)\Big\}\,\delta s \tag{4}$$

Using Appendix Equation (30) below the solution to Equation (4) above is...

$$\mathbb{E}\Big[V_t\Big] = C_0 \operatorname{Exp}\Big\{\mu\, t\Big\}\Big(\kappa - \mu\Big)^{-1} \tag{5}$$

We define the variable θ to be the cash flow valuation multiple. The equation for this multiple is...

$$\theta = \Big(\kappa - \mu\Big)^{-1} \text{ ...where... } \mu < \kappa \tag{6}$$

Using Equations (3) and (6) above we can rewrite expected company value Equation (5) above as...

$$\mathbb{E}\Big[V_t\Big] = \theta\, \mathbb{E}\Big[C_t\Big] = \theta\, C_0 \operatorname{Exp}\Big\{\mu\, t\Big\} \tag{7}$$

Note that the derivative of company value Equation (7) above with respect to time is...

$$\delta\mathbb{E}\Big[V_t\Big] = \mu\, \theta\, C_0 \operatorname{Exp}\Big\{\mu\, t\Big\}\,\delta t \tag{8}$$

The Evolution of Company Value Over Time

We will define the variable V_t to be random company value at time t. Using Equations (2) and (7) above the equation for random company value at time t is...

$$V_t = \theta\, C_t = \theta\, C_0 \,\mathrm{Exp}\left\{\left(\mu - \frac{1}{2}\,\sigma_c^2\right)t + \sigma_c \sqrt{t}\, x\right\} \tag{9}$$

We will define the variable ϕ to be the dividend yield. Given that total return equals the discount rate κ and per Equation (8) above the capital gains (stock price appreciation) rate is μ then the dividend yield must be the difference between those two rates. The equation for the dividend yield is...

$$\phi = \kappa - \mu \;\;\text{...such that...}\;\; \mu = \kappa - \phi \tag{10}$$

Using Equation (10) above we can rewrite Equation (9) above as...

$$V_t = \theta\, C_0 \,\mathrm{Exp}\left\{\left(\kappa - \phi - \frac{1}{2}\,\sigma_c^2\right)t + \sigma_c \sqrt{t}\, x\right\} \tag{11}$$

The equation for the natural log of Equation (11) above is...

$$\ln\left(V_t\right) = \ln\left(\theta\, C_0\right) + \left(\kappa - \phi - \frac{1}{2}\,\sigma_c^2\right)t + \sigma_c \sqrt{t}\, x \tag{12}$$

We will make the following variable definitions...

$$A = \ln\left(\theta\, C_0\right) + \left(\kappa - \phi - \frac{1}{2}\,\sigma_c^2\right)t \;\;\text{...and...}\;\; B = \sigma_c \sqrt{t} \tag{13}$$

Using Equation (13) above we will rewrite Equation (12) above as...

$$\ln\left(V_t\right) = A + B\, x \tag{14}$$

The equation for the first moment of Equation (14) above is...

$$\mathbb{E}\left[\ln\left(V_t\right)\right] = \mathbb{E}\left[A + B\, x\right] = A + B\,\mathbb{E}\left[x\right] = A \tag{15}$$

The equation for the second moment of Equation (14) above is...

$$\mathbb{E}\left[\ln\left(V_t\right)^2\right] = \mathbb{E}\left[A^2 + 2\,A\,B\,x + B^2 x^2\right] = A^2 + 2\,A\,B\,\mathbb{E}\left[x\right] + B^2\,\mathbb{E}\left[x^2\right] = A + B^2 \tag{16}$$

Using Equations (13) and (15) above the equation for the mean of Equation (14) above is...

$$\text{Mean of }\ln\left(V_t\right) = \text{First moment} = \ln\left(\theta\, C_0\right) + \left(\kappa - \phi - \frac{1}{2}\,\sigma_c^2\right)t \tag{17}$$

Using Equations (13), (15) and (16) above the equation for the variance of Equation (14) above is...

$$\text{Variance of }\ln\left(V_t\right) = \text{Second moment} - (\text{First moment})^2 = \sigma_c^2\, t \tag{18}$$

Houston, We Have A Problem

To quote from Robert Shiller's 1981 paper: We have seen that measures of stock price volatility over the past century appear to be far too high - five to thirteen times too high - to be attributed to new information about future real dividends. [1]

We defined the variable σ_c to be the volatility of cash flow. We will define the new variable σ_s to be the volatility of the log of historical stock prices. Per Dr. Shiller's quote above we can make the following observation...

$$\sigma_s > \sigma_c \text{ ...and... } 5 \leq \frac{\sigma_s}{\sigma_c} \leq 13 \tag{19}$$

We will define the variable σ_x to be the difference between the volatility of stock price and the volatility of the underlying cash flows. This statement in equation form is...

$$\sigma_x = \sigma_s - \sigma_c \text{ ...such that... } \sigma_s = \sigma_c + \sigma_x \tag{20}$$

To incorporate σ_x into our company valuation equation we will redefine the cash flow multiple (θ) to be a random variable. The new definition of theta is...

$$\text{Redefined cash flow mulitple: } \theta \operatorname{Exp}\left\{ -\frac{1}{2}\sigma_x^2\, t + \sigma_x \sqrt{t}\, y \right\} \text{ ...where... } y \sim N\Big[0, 1\Big] \tag{21}$$

The equation for the expected value of the random cash flow multiple as defined in Equation (21) above is...

$$\mathbb{E}\left[\theta \operatorname{Exp}\left\{ -\frac{1}{2}\sigma_x^2\, t + \sigma_x \sqrt{t}\, y \right\}\Bigg)\right] = \theta \text{ ...because... } \mathbb{E}\left[\operatorname{Exp}\left\{ -\frac{1}{2}\sigma_x^2\, t + \sigma_x \sqrt{t}\, y \right\}\right] = 1 \tag{22}$$

We will define the variable $\bar{V}_t$ to be redefined random company value at time t. Using Equations (11) and (21) above our redefined equation for company value becomes...

$$\bar{V}_t = \theta \operatorname{Exp}\left\{ -\frac{1}{2}\sigma_x^2\, t + \sigma_x \sqrt{t}\, y \right\} C_0 \operatorname{Exp}\left\{ \left(\kappa - \phi - \frac{1}{2}\sigma_c^2\right) t + \sigma_c \sqrt{t}\, x \right\} \tag{23}$$

Note that we can rewrite Equation (23) above as...

$$\bar{V}_t = \theta\, C_0 \operatorname{Exp}\left\{ \left(\kappa - \phi - \frac{1}{2}\sigma_c^2 - \frac{1}{2}\sigma_x^2\right) t + \sigma_c \sqrt{t}\, x + \sigma_x \sqrt{t}\, y \right\} \tag{24}$$

If the correlation between the diffusion component of the cash flow valuation multiple and the diffusion component of cash flow is zero then we can add the two variances together to get one variance. This statement in equation form is...

$$\sigma_s^2 = \sigma_c^2 + \sigma_x^2 \text{ ...such that... } \sigma_s = \sqrt{\sigma_c^2 + \sigma_x^2} \text{ ...given that... } \rho_{xy} = 0 \tag{25}$$

Using Equation (25) above we can rewrite company value Equation (24) above as...

$$\bar{V}_t = \theta\, C_0 \operatorname{Exp}\left\{ \left(\kappa - \phi - \frac{1}{2}\sigma_s^2\right) t + \sigma_s \sqrt{t}\, z \right\} \text{ ...where... } z \sim N\Big[0, 1\Big] \tag{26}$$

Note that the volatility parameter σ_s is directly observable using historical stock market data.

Note also that the expected value of Equation (26) above is...

$$\mathbb{E}\Big[\bar{V}_t\Big] = \theta\, \mathbb{E}\Big[C_t\Big] = \theta\, C_0 \operatorname{Exp}\Big\{\mu\, t\Big\} \tag{27}$$

References

[1] R. Shiller, *Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends?,* June, 1981.

Appendix

A. The solution to company value Equation (4) above is...

$$\begin{aligned} \mathbb{E}\Big[V_t\Big] &= \int_t^\infty C_0 \operatorname{Exp}\Big\{\mu\, s\Big\} \operatorname{Exp}\Big\{ -\kappa\,(s-t)\Big\}\,\delta s \\ &= C_0 \operatorname{Exp}\Big\{\kappa\, t\Big\} \int_t^\infty \operatorname{Exp}\Big\{(\mu - \kappa)\, s\Big\}\,\delta s \end{aligned} \tag{28}$$

The solution to Equation (28) above is...

$$\mathbb{E}\left[V_t\right] = C_0\,\text{Exp}\left\{\kappa\,t\right\}\frac{1}{\mu-\kappa}\Bigg[_{s=t}^{s=\infty}\text{Exp}\left\{(\mu-\kappa)\,s\right\}$$
$$= C_0\,\text{Exp}\left\{\kappa\,t\right\}\frac{1}{\mu-\kappa}\left[\text{Exp}\left\{(\mu-\kappa)\,\infty\right\}-\text{Exp}\left\{(\mu-\kappa)\,t\right\}\right] \tag{29}$$

If the continuous-time cash flow growth rate is less than the continuous-time discount rate then the solution to Equation (29) above is...

$$\mathbb{E}\left[V_t\right] = C_0\,\text{Exp}\left\{\kappa\,t\right\}\frac{1}{\mu-\kappa}\left[0-\text{Exp}\left\{(\mu-\kappa)\,t\right\}\right]$$
$$= C_0\,\text{Exp}\left\{\mu\,t\right\}\left(\kappa-\mu\right)^{-1} \tag{30}$$